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T-Points: A Codimension Two Heteroclinic Bifurcation

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The local bifurcation structure of a heteroclinic bifurcation which has been observed in the Lorenz equations is analyzed. The existence of a particular heteroclinic loop at one point in a two-dimensional parameter space (a "*T* point") implies the existence of a line of heteroclinic loops and a logarithmic spiral of homoclinic orbits, as well as countably many other topologically more complicated *T* points in a small neighborhood in parameter space.

KEY WORDS: Homoclinic orbit; Heteroclinic orbit; bifurcations; chaos; Lorenz equations.

1. INTRODUCTION

The Lorenz equations[5,6]

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

have been studied by many authors over the last 20 years, but are still a rich source of novel behavior. A good understanding of many of the complicated bifurcation phenomena which have been observed in these equations can be obtained by an analysis of the homoclinic orbits and heteroclinic loops which occur.[3,6] (A homoclinic orbit is a trajectory which tends to the same stationary point as $t \to \pm\infty$ and a heteroclinic orbit is a trajectory which tends to two different stationary points, one as

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$t \to -\infty$ and the other as $t \to \infty$). A heteroclinic loop is a set of heteroclinic orbits between stationary points A_1 and A_2, A_2 and $A_3, \ldots, A_{n-1}$ and A_n, and finally A_n and A_1. The Lorenz equations are invariant under the symmetry

$$(x, y, z) \to (-x, -y, z) \tag{2}$$

and, over a large region of parameter space, there are three unstable stationary points: the origin, 0, and $C^{\pm} = [\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1]$ which are mapped onto each other by the symmetry. In a two-parameter numerical study[1] in which σ was kept fixed, $\sigma = 10$, it has been shown that there are particular parameter values ($r \cong 30.475$, $b \cong 2.623$) for which each branch of the unstable manifold of the origin coincides with a branch of the stable manifold of C^+ or C^- giving the configuration shown schematically in Fig. 1. Such points in a two-dimensional parameter space will be referred to as *T points*. This follows Ref. 1 where such a point was referred to as a terminal point, but which could also stand for codimension two point.

We look at the simple heteroclinic and homoclinic orbits in a neighborhood of a T point. Although the results will be phrased in terms of systems with symmetry like the Lorenz equations, many remain true (with minor reinterpretation, see Ref. 3) for any family of systems with a

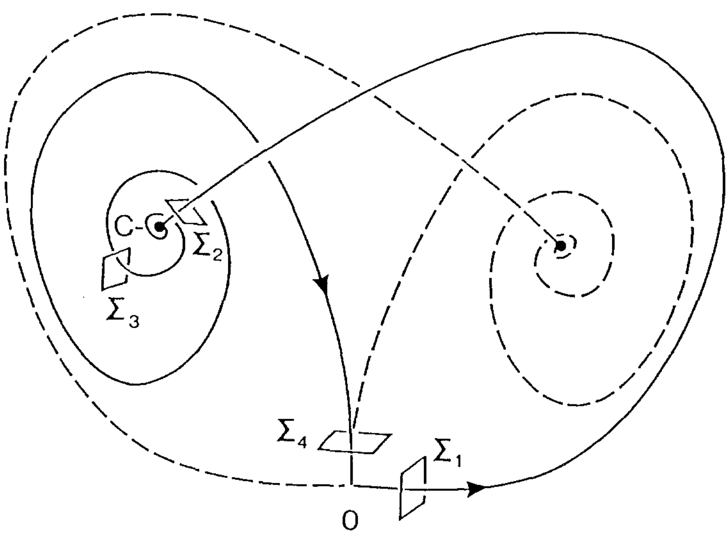

Fig. 1. The heteroclinic loop, showing the surfaces Σ_1, Σ_2, Σ_3, and Σ_4 used to construct the return map S. The symmetric image of this loop is shown with a broken line.

codimension two heteroclinic loop between two stationary points with the same linearized flow as 0 and $C^{\pm}$. The existence of horseshoes in this more general situation was proved in Ref. 9.

2. A RETURN MAP

At parameter values near the T point, trajectories which pass close to 0 follow the unstable manifold and are reinjected into a neighborhood of one of the other stationary points. Likewise, some trajectories which start close to C^+ or C^- follow the trajectories shown in Fig. 1 and hence return to a neighborhood of 0. Standard techniques can therefore be used to derive a return map on a plane near 0: the flow in a sufficiently small neighborhood of the stationary points is taken to be linear, and can be solved explicitly, while the global reinjection along the unstable manifolds is modeled by affine transformations. This procedure provides a first-order analysis in a small neighborhood of the heteroclinic orbits and stationary points for parameter values close to the T point. The results we obtain will hold (in some undetermined small neighborhoods) for the Lorenz equations and, in general, for systems satisfying the hypotheses below.[9,11]

Figure 1 shows the invariant manifolds of the stationary points 0 and $C^{\pm}$. The positive branch of the unstable manifold of 0 (which leaves 0 into $x>0$) tends to C^- at the T-point parameter values, and so, by symmetry, the negative branch of the unstable manifold approaches C^+. We concentrate on the heteroclinic loop between 0 and C^-, formed by the positive branch of the unstable manifold of 0 and the trajectory from C^- back to 0. We compute a return map on a small section of a plane $\Sigma_4 = \{(x, y, z): z = h_2, h_2 \text{ small}\}$ just above 0 and shown in Fig. 1. This return map will be valid for points on Σ_4 lying to the right of the stable manifold of the origin (i.e., for trajectories which pass downward through Σ_4 before leaving the neighborhood of 0 close to the positive branch of the unstable manifold) and this part of the analysis will be valid for any heteroclinic loop between two stationary points with the properties described below. In Section 3 we will use the symmetry, (2), to derive some properties of flows at parameter values near the T point which involve trajectories which pass close to both C^+ and C^-.

We begin by dividing the flow into four parts and derive maps T_i, $i = 1, 2, 3, 4$ where $T_1: \Sigma_4 \to \Sigma_1$, $T_2: \Sigma_1 \to \Sigma_2$, $T_3: \Sigma_3 \to \Sigma_2$, and T_4: $\Sigma_3 \to \Sigma_4$, where the four planes Σ_i are as shown in Fig. 1 and described below. The maps T_1 and T_3 depend only on the flow near the stationary points, which is taken to be linear, whereas T_2 and T_4 reflect global properties of the flow between the stationary points. It is convenient (and natural, as we see below) to assume that all the parameter dependence is in the map

T_2 and that T_1, T_3, and T_4 do not depend on the parameters. Thus, in particular, we assume that the eigenvalues of the linearized flow near 0 and $C^{\pm}$ do not depend on the parameters. This assumption is also justified in the context of our first-order analysis. We choose coordinates (x, y, z) near 0 [different from the original (x, y, z) in (1)] and (X, Y, Z) near $C^{\pm}$ such that the flow in a neighborhood of 0 and $C^{\pm}$ can be written as

$$\begin{aligned} \dot{X} &= PX - \Omega Y & & \qquad \dot{x} = \lambda_1 x \\ \dot{Y} &= \Omega X + PY \qquad \text{at} \quad C^{\pm} & & \qquad \dot{y} = -\lambda_2 y \qquad \text{at} \quad 0 \\ \dot{Z} &= -\Lambda Z & & \qquad \dot{z} = -\lambda_3 z \end{aligned}$$

where $\Lambda > P > 0$ and $\lambda_2 > \lambda_1 > \lambda_3 > 0$. These inequalities are certainly satisfied by the Lorenz equations for parameter values near the T point. In these coordinates the planes Σ_i can be chosen to be

$$\begin{aligned} \Sigma_1 &= \{(x, y, z) \mid x = h_1\} \\ \Sigma_2 &= \{(X, Y, Z) \mid Z = H\} \\ \Sigma_3 &= \{(X, Y, Z) \mid Y = 0\} \\ \Sigma_4 &= \{(x, y, z) \mid z = h_2\} \end{aligned}$$

although when constructing the return map Σ_3 must be restricted so that trajectories only strike the surface once as they spiral out of $C^{\pm}$. This allows us to write the maps $T_1: \Sigma_4 \to \Sigma_1$ and $T_3: \Sigma_3 \to \Sigma_2$ explicitly:

$$T_1(x, y, h_2) = (h_1, y', z'), \qquad \text{where} \quad x > 0 \quad \text{and} \quad \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} pyx^{\lambda_2/\lambda_1} \\ qx^{\delta} \end{pmatrix} \tag{3}$$

$$T_3(X, 0, Z) = (X', Y', H), \qquad \text{where} \quad \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} XZ^{\Delta} \cos(\Xi \ln Z + \Phi) \\ XZ^{\Delta} \sin(\Xi \ln Z + \Phi) \end{pmatrix} \tag{4}$$

Here $\delta = \lambda_3/\lambda_1$, $\Delta = P/\Lambda$, $\Xi = -\Omega/\Lambda$, and p, q, and Φ are constants. Notice that the direction of the flow is from Σ_2 to Σ_3, and we compute T_3 instead of T_3^{-1} for ease of analysis.

Now consider the map $T_4: \Sigma_3 \to \Sigma_4$. The local unstable manifold of C^- is two-dimensional and intersects Σ_3 with $Z = 0$. The stable manifold of 0 is also two-dimensional and intersects Σ_4 with $x = 0$. Since two-dimensional manifolds intersect generically in $\mathbb{R}^3$, we assume that the unstable manifold of C^- and the stable manifold of 0 intersect along a single trajectory. Thus we assume that there is always a heteroclinic connection between C^- and 0. If we write $T_4(X, 0, Z) = (x, y, h_2)$ where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix} \tag{5}$$

with α, β, A, B, C, and D constants, and the matrix is nonsingular, the condition that there is always a heteroclinic connection between C^- and 0 is equivalent to saying that there is a point $(X, 0, 0)$ on Σ_3 which is mapped to $(0, y, h_2)$ on Σ_4 for some y. This implies that

$$\alpha + AX = 0 \tag{6}$$

for some X with $(X, 0, 0)$ on Σ_3.

In constructing the map T_2 it is natural to introduce two parameters, μ and ν, which give the (X, Y) coordinates on Σ_2 where the unstable manifold of 0 strikes Σ_2 for the first time. Since our choice of coordinates near C^- implies that the (one-dimensional) local stable manifold of C^- intersects Σ_2 at $(0, 0, H)$, we will have a heteroclinic connection (and hence the T point) between 0 and C^- when $(\mu, \nu) = (0, 0)$. Thus, to first order, we write $T_2(h_1, y, z) = (X, Y, H)$ where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu \\ \nu \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{7}$$

This is permissible at first order since the time spent traveling between Σ_1 and Σ_2 is small compared with the time spent near the stationary points. The constant matrix with elements a, b, c, and d is assumed to be nonsingular.

The full return map $S: \Sigma_4 \to \Sigma_4$, defined for $x > 0$, is $T_4 \circ T_3^{-1} \circ T_2 \circ T_1$. We do not study S here because it is too unwieldy (T_3^{-1} is particularly nasty) but we can use T_1, T_2, T_3, and T_4 to find simple homoclinic and heteroclinic orbits near the T point $(\mu, \nu) = (0, 0)$.

3. HOMOCLINIC ORBITS AND SUBSIDIARY *T* POINTS

Figure 2(a) shows an orbit homoclinic to 0 which passes close to C^-. For this to occur, the unstable manifold of the origin must, after passing close to C^-, strike Σ_4 with $x = 0$. So, using T_4

$$\alpha + AX + BZ = 0 \tag{8}$$

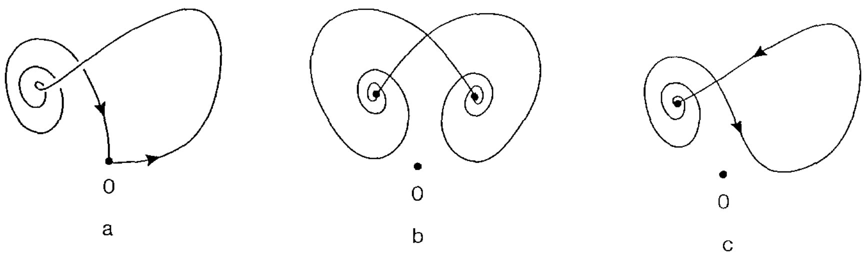

Fig. 2. Schematic drawings of (a) a homoclinic orbit to the origin, (b) a heteroclinic loop between C^+ and C^-, (c) a homoclinic orbit to C^+. Note that it does not pass close to C^-.

where $(X, 0, Z)$ is the coordinate of the intersection of the unstable manifold of 0 with Σ_3. Thus, from T_3

$$\begin{aligned} \mu &= XZ^{\Delta} \cos(\Xi \ln Z + \Phi) \\ v &= XZ^{\Delta} \sin(\Xi \ln Z + \Phi) \end{aligned} \tag{9}$$

Hence

$$\begin{aligned} \mu &= -\frac{1}{A}(\alpha + BZ)\, Z^{\Delta} \cos(\Xi \ln Z + \Phi) \\ v &= -\frac{1}{A}(\alpha + BZ)\, Z^{\Delta} \sin(\Xi \ln Z + \Phi) \end{aligned} \tag{10}$$

As $Z \to 0$, (10) describes a logarithmic spiral in the (μ, v) plane, so there are homoclinic orbits of the form shown schematically in Fig. 2(a) along this spiral which tends to the T point $(0, 0)$ in parameter space (see Fig. 3). For any generic one-parameter family of systems which passes through the T point there are infinite sequences of such homoclinic orbits (on both sides of the T point) which accumulate to the T point at the rate $\exp(-2\pi P/\Omega)$.

To find the locus in parameter space of the simplest heteroclinic loop between C^+ and C^- we can use the symmetry of the Lorenz equations. If there is a point (x, y) with $x > 0$ on Σ_4 which lies on the stable manifold of C^- and, at the same parameter values $(-x, -y) \in \Sigma_4$ lies on the unstable manifold of C^-, then there must be a heteroclinic connection of the form shown in Fig. 2(b).

The first condition implies $T_2[T_1(x, y, h_2)] = (0, 0, H)$ and therefore

$$\begin{aligned} 0 &= \mu + apyx^{\lambda_2/\lambda_1} + bqx^{\delta} \\ 0 &= v + cpyx^{\lambda_2/\lambda_1} + dqx^{\delta} \end{aligned} \tag{11}$$

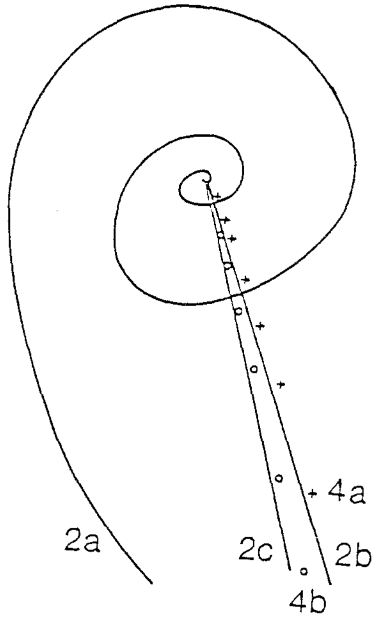

Fig. 3. The local bifurcation picture. The labels give the figures showing the form of the heteroclinic loop or homoclinic orbit which occurs along the relevant line or set of points.

The second implies that there exists an X such that $T_4(X, 0, 0) = (-x, -y, h_2)$ and therefore

$$\begin{aligned} -x &= \alpha + AX < 0 \\ -y &= \beta + CX \end{aligned} \tag{12}$$

for some X. Hence

$$\begin{aligned} 0 &\approx \mu + bq\,|\alpha + AX|^{\delta} \\ 0 &\approx \nu + dq\,|\alpha + AX|^{\delta} \end{aligned} \tag{13}$$

Retaining only the leading order terms gives $\nu \approx d\mu/b$, with $\operatorname{sgn} \mu = -\operatorname{sgn} K_1$, $K_1 = bq$. Thus there is a half-line of heteroclinic orbits which terminates at the T point $(\mu, \nu) = (0, 0)$, given by $\nu \approx d\mu/b$, $\operatorname{sgn} \mu = \text{constant}$. These heteroclinic orbits have been discussed in Ref. 3 for parameter values outside a small neighborhood of the T point: if $\Delta < 1$ there are lines of heteroclinic and homoclinic orbits (involving only the stationary points C^+ and C^-) which accumulate on this line. In a neighborhood of the T point it is also possible to deduce the existence of other homoclinic orbits. For example, replacing $(-x, -y, h_2)$ by (x, y, h_2)

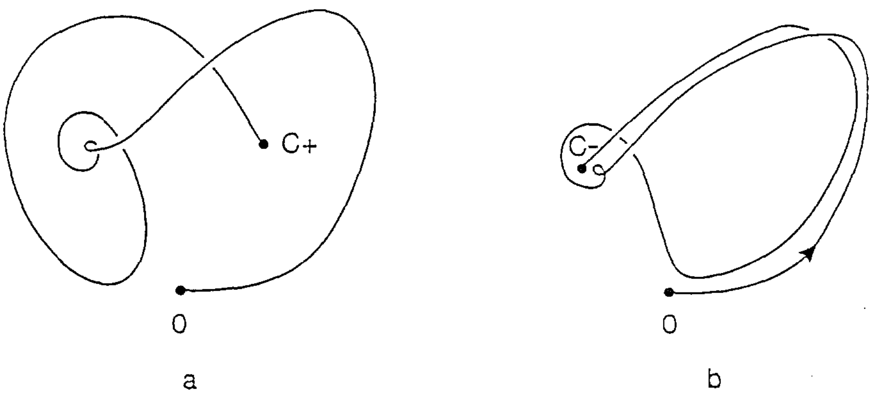

Fig. 4. One branch of the unstable manifold of the origin for the subsidiary T points shown in Fig. 3.

on Σ_4 in the analysis above, we find a half-line of homoclinic orbits to $C^{\pm}$ of the form shown schematically in Fig. 2(c). To lowest order these also lie on the half-line $v = d\mu/b$, $\operatorname{sgn} \mu = -\operatorname{sgn} K_1$, although higher order terms are different: the equivalent of (13) are obtained by changing the signs of a and c.

Figure 3 shows the theoretical bifurcation diagram near the T point. It agrees with the numerically computed picture in Ref. 1, except that it contains additional bifurcations, and in Ref. 1 the logarithmic spiral of homoclinic orbits was mistakenly identified as a series of concentric circles, probably for complicated reasons to do with interaction between the topology of the homoclinic orbits and the computer program used to compute them.

Obviously there are many more complicated homoclinic and heteroclinic orbits which we have not discussed but which could, in principle, be analyzed using the return map S and added to Fig. 3. In particular, there are also subsidiary T points which accumulate at the principal T point $(0, 0)$. For example, using the return map we can look for T points where the connections are as shown in Fig. 4. Some horrendous algebra leads one to the conclusion that there are sequences of such T-points which approach $(0, 0)$ asymptotically along the half-line $v = d\mu/b$, $\operatorname{sgn} \mu = -\operatorname{sgn} K_1$ (see Fig. 3). Each of these subsidiary T points is amenable to the same analysis described above for the principal T point (although with a smaller neighborhood) and, in particular, there are sequences of T points which tend to these T points. And so on.

4. FINAL REMARKS

In previous papers[2,3,6,7] we argued that some details of complicated sequences of bifurcations occurring in ordinary differential equations can be understood by keeping careful track of the periodic orbits which are created and destroyed in the bifurcations associated with homoclinic orbits and heteroclinic loops. For the bifurcations discussed above, homoclinic orbits to the origin have been analyzed in Ref. 6, and in Ref. 3 we suggested that it was reasonable, in certain circumstances, to think of the bifurcation associated with the symmetric heteroclinic loop between C^+ and C^- as producing a single symmetric periodic orbit, despite the complicated sequence of bifurcations which may actually occur. (If $\Delta < 1$, as is the case for the Lorenz equations, a more complicated sequence of bifurcations is assured.) At first sight, this argument appears to produce a contradiction when one considers a circular path through the two-dimensional parameter space which encloses the T point. At a particular point on the circular path there is a fixed number of periodic orbits in the system. Now consider moving once around the circle in parameter space; the heteroclinic line is crossed only once, resulting in the production of a periodic orbit which is not obviously destroyed elsewhere on the path. However, any circular path will necessarily cross the spiral of homoclinic orbits to the origin an odd number of times and it is possible to resolve the contradiction by showing that the odd homoclinic bifurcation results in the destruction of the orbit produced in the heteroclinic bifurcation. This type of argument actually has many implications for details of the local bifurcation picture which are not easily accessible via rigorous analysis of the return map S. Furthermore, similar considerations give considerable insight into global bifurcation patterns in large regions of (r, b)-space in the Lorenz equations. A full discussion of the global results will be published elsewhere. For the moment, notice that the most frequently studied parameter values, $0 < r < \infty$, $b = 8/3$, $\sigma = 10$, gives a line in (r, b)-parameter space which passes very close to the T point mentioned at the beginning of this paper. The logarithmic spiral of homoclinic orbits to the origin [equation (10) and Fig. 2(a)] intersects this line for r values close to 30. This fact can be used to explain, for the first time, why the Lorenz system ($b = 8/3$, $\sigma = 10$) changes from the regime in which the geometric Lorenz attractor[4,10] appears to be a good model of the behavior (in $r < 28$ and before the spiral is crossed) through a parameter interval in which a "twist" is introduced into the flow and a hook into appropriate return maps ($28 < r < 31$, see Ref. 6), to the regime in which stable periodic orbits and period-doubling bifurcations occur ($r > 31$ and after the spiral has been crossed).

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